The Coble hypersurfaces

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Introduction

The title refers to the following nice observations of Coble. Let A be a complex abelian variety, of dimension g, and $\mathcal{L}$ a line bundle on A defining a principal polarization (that is, $\mathcal{L}$ is ample and $\dim \mathrm{H}^0(\mathrm{A},\mathcal{L})=1$). We will assume throughout that $(\mathrm{A},\mathcal{L})$ is *indecomposable*, that is, cannot be written as a product of principally polarized abelian varieties of lower dimension.

Fix an integer $\nu \geq 1$ and put $\mathrm{V}_\nu = \mathrm{H}^0(\mathrm{A},\mathcal{L}^\nu)$. We consider the morphism[1] $\varphi_\nu : \mathrm{A} \rightarrow \mathbb{P}(\mathrm{V}_\nu)$ defined by the global sections of $\mathcal{L}^\nu$. Recall that φ_ν is an embedding for $\nu \geq 3$, and that φ_2 induces an embedding of the Kummer variety $\mathrm{A}/\{\pm 1\}$ in $\mathbb{P}(\mathrm{V}_2)$. Let A_ν be the kernel of the multiplication by ν in A; the group A_ν acts on A and on $\mathbb{P}(\mathrm{V}_\nu)$ in such a way that φ_ν is A_ν-equivariant.

Proposition (Coble) .– 1) *Let $g=2$. There exists a unique A_3-invariant cubic hypersurface in $\mathbb{P}(\mathrm{V}_3)$ $(\cong \mathbb{P}^8)$ that is singular along $\varphi_3(\mathrm{A})$. The polars of this cubic span the space of quadrics in $\mathbb{P}(\mathrm{V}_3)$ containing $\varphi_3(\mathrm{A})$.*

2) *Let $g=3$. There exists a unique A_2-invariant quartic hypersurface in $\mathbb{P}(\mathrm{V}_2)$ $(\cong \mathbb{P}^7)$ that is singular along $\varphi_2(\mathrm{A})$. The polars of this quartic span the space of cubic hypersurfaces in $\mathbb{P}(\mathrm{V}_2)$ containing $\varphi_2(\mathrm{A})$.*

The proof of 2) appears in [C2], and that of 1) in [C1] (actually the cubic is not explicitly mentioned in that paper, but it is easily deduced from the equations for the quadrics containing $\varphi_3(\mathrm{A})$. I am indebted to I. Dolgachev for this reference). Both results are proved by explicit computations. These hypersurfaces have a beautiful interpretation in terms of vector bundles on curves (see [N-R] for the quartic and [O] for the cubic).

An analogous statement appears in [O-P], this time for the moduli space $\mathcal{SU}_{\mathrm{C}}(2)$ of semi-stable rank 2 vector bundles with trivial determinant on a non-hyperelliptic curve of genus 4 (this moduli space is naturally embedded in $\mathbb{P}(\mathrm{V}_2)$). Oxbury and Pauly prove that it is contained in a unique A_2-invariant quartic hypersurface, whose polars span the space of cubic hypersurfaces containing $\mathcal{SU}_{\mathrm{C}}(2)$.

The main observation of this note is that these facts follow from a general (and elementary) result about representations of the Heisenberg group (Proposition 2.1 below). Let us just mention here a geometric consequence of that result:

Proposition .– *Let $n=3$ or 4; put $\nu=3$ if $n=3$, $\nu=2$ if $n=4$. Let $(\mathrm{T}_1,\ldots,\mathrm{T}_{\mathrm{N}})$ be a coordinate system on $\mathbb{P}(\mathrm{V}_\nu)$. Let X be an A_ν-invariant subvariety of $\mathbb{P}(\mathrm{V}_\nu)$. Then the space of hypersurfaces of degree $n-1$ containing X*

[1] We use Grothendieck's notation: $\mathbb{P}(\mathrm{V}_\nu)$ is the space of hyperplanes of V_ν.

admits a basis $(\partial \mathrm{F}_i/\partial \mathrm{T}_j)$, *where* $\mathrm{F}_1,\ldots,\mathrm{F}_m$ *are forms of degree* n *on* $\mathbb{P}(\mathrm{V}_\nu)$, *such that the hypersurfaces* $\mathrm{F}_i=0$ *are* A_ν*-invariant (and singular along* X).

1. Heisenberg submodules of $\mathsf{S}^{n-1}\mathrm{V}$

Let n be an integer; we put $\nu=n$ if n is odd, $\nu=n/2$ if n is even. We write for brevity V instead of V_ν . We will occasionally pick a coordinate system $(\mathrm{T}_1,\ldots,\mathrm{T}_\mathrm{N})$ on $\mathbb{P}(\mathrm{V}_\nu)$, to make some of our statements more concrete.

The action of A_ν on $\mathbb{P}(\mathrm{V})$ lifts to an action on V of a central extension $\widetilde{\mathrm{A}}_\nu$ of A_ν by $\mathbb{C}^*$. For all $\gamma\in\widetilde{\mathrm{A}}_\nu$, the element γ^n belongs to the center $\mathbb{C}^*$ of $\widetilde{\mathrm{A}}_\nu$, and the map $\gamma\mapsto\gamma^n$ is a homomorphism of $\widetilde{\mathrm{A}}_\nu$ onto $\mathbb{C}^*$ (this is where we need to take $n=2\nu$ instead of ν when ν is even). We denote by H_n its kernel; it is a central extension

$$1\rightarrow\boldsymbol{\mu}_n\longrightarrow\mathrm{H}_n\longrightarrow\mathrm{A}_\nu\rightarrow 0$$

of A_ν by the group $\boldsymbol{\mu}_n$ of n-th roots of unity in $\mathbb{C}$.

Proposition 1 .– *Assume* $n=3$ *or* 4 *. Let* W *be an irreducible sub-*H_n*-module of* $\mathsf{S}^{n-1}\mathrm{V}$ *. There exists a* H_n*-invariant form* $\mathrm{F}\in\mathsf{S}^n\mathrm{V}$ *, unique up to a scalar, such that* $(\partial\mathrm{F}/\partial\mathrm{T}_1,\ldots,\partial\mathrm{F}/\partial\mathrm{T}_\mathrm{N})$ *form a basis of* W *.*

Proof : Put $\mathrm{N}=\dim\mathrm{V}\ (=\nu^g)$. The group H_n acts irreducibly on V , and this is the unique irreducible representation of H_n on which the center $\boldsymbol{\mu}_n$ acts by homotheties. It follows that the representation of H_n on $\mathsf{S}^{n-1}\mathrm{V}$ is isomorphic to the direct sum of k copies of V^* , with

$$k=\dim\mathsf{S}^{n-1}\mathrm{V}/\dim\mathrm{V}^*=\frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1}\ .$$

The space $\mathrm{Hom}_{\mathrm{H}_n}(\mathrm{V}^*,\mathsf{S}^{n-1}\mathrm{V})$ has dimension k ; it parametrizes the irreducible sub-H_n-modules of $\mathsf{S}^{n-1}\mathrm{V}$.

Consider the H_n-equivariant injective map

$$h:\mathsf{S}^n\mathrm{V}\longrightarrow\mathrm{Hom}(\mathrm{V}^*,\mathsf{S}^{n-1}\mathrm{V})$$

given by $h(\mathrm{F})(\partial)=\partial\mathrm{F}$ (we identify V^* with the space of degree -1 derivations of $\mathsf{S}\mathrm{V}$). It induces an injection $(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n}\hookrightarrow\mathrm{Hom}_{\mathrm{H}_n}(\mathrm{V}^*,\mathsf{S}^{n-1}\mathrm{V})$ of the H_n-invariant subspaces. The assertion of the Proposition is that this map is onto, or equivalently that $\dim(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n}=\frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1}$.

The action of H_n on $\mathsf{S}^n\mathrm{V}$ factors through the abelian quotient A_ν , hence is the direct sum of 1-dimensional representations V_χ corresponding to characters χ of A_ν . We claim that all non-trivial characters of A_ν appear with the same multiplicity. To see this, consider the group $\mathrm{Aut}(\mathrm{H}_n,\boldsymbol{\mu}_n)$ of automorphisms of H_n which induce the identity on $\boldsymbol{\mu}_n$. Because of the unicity property of the

representation $\rho : \mathrm{H}_n \to \mathrm{GL}(\mathrm{V})$, for every $\varphi \in \mathrm{Aut}(\mathrm{H}_n, \boldsymbol{\mu}_n)$ the representation $\rho\circ\varphi$ is isomorphic to ρ, thus $(\mathsf{S}^n\rho)\circ\varphi$ is isomorphic to $\mathsf{S}^n\rho$. This implies that the characters appearing in the decomposition of $\mathsf{S}^n\mathrm{V}$ are exchanged by the action of $\mathrm{Aut}(\mathrm{H}_n, \boldsymbol{\mu}_n)$. But the action of $\mathrm{Aut}(\mathrm{H}_n, \boldsymbol{\mu}_n)$ on A_ν factors through a surjective homomorphism $\mathrm{Aut}(\mathrm{H}_n, \boldsymbol{\mu}_n) \to \mathrm{Sp}(\mathrm{A}_\nu)$ (see e.g. [B-L], ch. 6, lemma 6.6). Since ν is prime, the symplectic group $\mathrm{Sp}(\mathrm{A}_\nu)$ acts transitively on the set of nontrivial characters of A_ν, hence our claim.

Thus we have

$$\mathsf{S}^n\mathrm{V} = \Big(\bigoplus_{\chi\neq 1} \mathrm{V}_\chi\Big)^m \oplus (\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n}$$

for some integer $m \geq 0$. Counting dimensions yields

$$\binom{\mathrm{N}+n-1}{n} = m\,(\mathrm{N}^2-1) + \dim(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n}\ .$$

On the other hand a simple computation gives

$$\binom{\mathrm{N}+n-1}{n} = m\,(\mathrm{N}^2-1) + \frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1}\ ,$$

with $m = \frac{1}{6}(\mathrm{N}+3)$ for $n=3$, and $m = \frac{1}{24}(\mathrm{N}+2)(\mathrm{N}+4)$ for $n=4$. Moreover we have $\dim(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n} \leq \frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1} < \mathrm{N}^2-1$. Thus $\dim(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n}$ and $\frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1}$ are both equal to the rest of the division of $\binom{\mathrm{N}+n-1}{n}$ by N^2-1, hence they are equal. ■

Corollary.– *Let* X *be a subvariety of* $\mathbb{P}(\mathrm{V})$*, invariant under the action of* A_ν*; denote by* $\mathcal{I}_\mathrm{X}$ *the ideal sheaf of* X *in* $\mathbb{P}(\mathrm{V})$*. Let* $(\mathrm{F}_1,\ldots,\mathrm{F}_m)$ *be a basis of the space of* H_n*-invariant forms in* $\mathsf{S}^n\mathrm{V}$ *which are singular along* X*. Then the partial derivatives* $(\partial\mathrm{F}_i/\partial\mathrm{T}_j)$ *form a basis of* $\mathrm{H}^0(\mathbb{P}(\mathrm{V}),\mathcal{I}_\mathrm{X}(n-1))$*. In particular, if* $\dim \mathrm{H}^0(\mathbb{P}(\mathrm{V}),\mathcal{I}_\mathrm{X}(n-1)) = \nu^g$*, there exists a unique* H_n*-invariant form in* $\mathsf{S}^n\mathrm{V}$ *which is singular along* X*.*

Indeed $\mathrm{H}^0(\mathbb{P}(\mathrm{V}),\mathcal{I}_\mathrm{X}(n-1))$ is a sub-H_n-module of $\mathrm{H}^0(\mathbb{P}(\mathrm{V}),\mathcal{O}_\mathbb{P}(n-1)) = \mathsf{S}^{n-1}\mathrm{V}$, and therefore isomorphic to a direct sum of simple modules. ■

In the next section we will apply the Corollary to the abelian variety A embedded in $\mathbb{P}(\mathrm{V}_\nu)$. Another interesting case is when X is the moduli space of vector bundles of rank 2 and trivial determinant on a curve C of genus 3 with no vanishing theta-constant. Let A be the Jacobian of C; then X has a natural A_2-equivariant embedding in $\mathbb{P}(\mathrm{V}_2)$, and Oxbury and Pauly prove the equality $\dim \mathrm{H}^0(\mathbb{P}(\mathrm{V}_2),\mathcal{I}_\mathrm{X}(3)) = 8$ [O-P]. Therefore there exists a unique H_4-invariant quartic hypersurface singular along X.

Remark.– Unfortunately the cases $n=3$ and $n=4$ seem to be the only ones for which the Proposition holds. If for instance n is prime ≥ 5, it is easy to check that the equality $\dim(\mathsf{S}^n\mathrm{V})^{\mathrm{H}_n} = \frac{1}{\mathrm{N}}\binom{\mathrm{N}+n-2}{n-1}$ *never* holds.

2. Application: equations for abelian varieties

(2.1) Let us apply the Corollary to $\mathrm{X}=\varphi_\nu(\mathrm{A})$ embedded in $\mathbb{P}(\mathrm{V}_\nu)$. If $n=4$ we will assume that $(\mathrm{A},\mathcal{L})$ *has no vanishing theta-constant* (that is, no symmetric theta divisor singular at 0 – if $g=3$ this simply means that $(\mathrm{A},\mathcal{L})$ is the Jacobian of a non-hyperelliptic curve). This implies that the Kummer variety $\varphi_2(\mathrm{A})\subset\mathbb{P}(\mathrm{V}_2)$ is projectively normal, while $\varphi_3(\mathrm{A})$ is always projectively normal in $\mathbb{P}(\mathrm{V}_3)$ [Ko]. Thus the natural map $\mathrm{H}^0(\mathbb{P}(\mathrm{V}_\nu),\mathcal{O}_{\mathbb{P}}(n-1))\to\mathrm{H}^0(\mathrm{X},\mathcal{O}_{\mathrm{X}}(n-1))$ is surjective, and this allows us to compute the dimension of its kernel. We find that *the space of* H_n*-invariant forms in* $\mathsf{S}^n\mathrm{V}$ *singular along* X *has dimension* $m_n(g)$ *given by*

$$m_3(g)=\frac{1}{2}(3^g-2^{g+1}+1)\qquad m_4(g)=\frac{1}{6}(2^g(2^g+3)-3^{g+1}-1)\ ;$$

for any basis $(\mathrm{F}_1,\ldots,\mathrm{F}_{m_n(g)})$ of this space, the derivatives $(\partial\mathrm{F}_i/\partial\mathrm{T}_j)$ form a basis of the space of forms of degree $n-1$ vanishing along X.

(2.2) Let us consider in particular the case $g=n-1$ considered by Coble. Since $m_3(2)=m_4(3)=1$ we recover Coble's result: there is a unique H_n-invariant hypersurface of degree n singular along $\varphi_\nu(\mathrm{A})$. In fact we have a slightly better result:

Proposition 2.– *Assume* $g=n-1$. *The Coble hypersurface in* $\mathbb{P}(\mathrm{V}_\nu)$ *is the unique hypersurface of degree* n *singular along* $\varphi_\nu(\mathrm{A})$.

Proof: The case of the Coble quartic is explained in [L], and the proof works equally well for the cubic. Let us recall briefly the argument. Let $\mathrm{F}=0$ be the Coble hypersurface. The derivatives $\partial\mathrm{F}/\partial\mathrm{T}_1,\ldots,\partial\mathrm{F}/\partial\mathrm{T}_\mathrm{N}$ span the space I_{n-1} of forms of degree $n-1$ vanishing along $\varphi_\nu(\mathrm{A})$; the action of H_n on I_{n-1} is irreducible.

Let W be the space of forms of degree n which are singular along $\varphi_\nu(\mathrm{A})$; it is a sub-H_n-module of $\mathsf{S}^n\mathrm{V}$, hence a sum of one-dimensional representations W_χ. Let $\mathrm{G}\neq 0$ in W_χ. The derivatives $\partial\mathrm{G}/\partial\mathrm{T}_1,\ldots,\partial\mathrm{G}/\partial\mathrm{T}_\mathrm{N}$ vanish on $\varphi_\nu(\mathrm{A})$, hence span a subspace of I_{n-1}; since this subspace is stable under H_n, it is equal to I_{n-1}. By [D, § 1] this implies that there exists an automorphism T of V_ν such that $\mathrm{G}=\mathrm{F}\circ\mathrm{T}$.

Now the singular locus of the Coble hypersurface is exactly $\varphi_\nu(\mathrm{A})$ (see (2.3) below); thus T must preserve $\varphi_\nu(\mathrm{A})$. In the group of automorphisms of V_ν preserving $\varphi_\nu(\mathrm{A})$, the Heisenberg group H_n is normal – because the group of translations of A is normal inside the group of all automorphisms. Thus T normalizes H_n; this implies that the form $\mathrm{G}=\mathrm{F}\circ\mathrm{T}$ is H_n-invariant, and therefore proportional to F by Coble's result. ∎

(2.3) For $g=2$, Coble states in [C1] that $\varphi_3(\mathrm{A})$ is the set-theoretical intersection of the quadrics that contain it – in other words, $\varphi_3(\mathrm{A})$ is the singular locus of the Coble cubic; this is proved even scheme-theoretically in [B]. When $g=3$ and

$(\mathrm{A}, \mathcal{L})$ has no vanishing theta-constant, Narasimhan and Ramanan have proved that the Kummer variety $\varphi_2(\mathrm{A})$ is set-theoretically the singular locus of the Coble quartic [N-R]; this holds also scheme-theoretically by [L]. It is tempting to conjecture that both statements hold in higher dimension as well, namely that the abelian variety $\varphi_3(\mathrm{A})$ is a scheme-theoretical intersection of quadrics and that the Kummer variety $\varphi_2(\mathrm{A})$ is a scheme-theoretical intersection of cubics. Note, however, that these quadrics or cubics cannot generate the full ideal of $\varphi_\nu(\mathrm{A})$:

Proposition 3 .– *The graded ideal* I *of* $\varphi_\nu(\mathrm{A})$ *in* $\mathbb{P}(\mathrm{V}_\nu)$ *is* not *generated by its elements of degree* $\leq n-1$.

(Recall that I is generated by its elements of degree $\leq n$, see [B-L], ch. 7 and [K].)

Note that the Proposition is immediate in the case $g = n-1$ considered by Coble, because then $\dim(\mathrm{V}\otimes \mathrm{I}_{n-1}) < \dim \mathrm{I}_n$. However this inequality does not hold any more in higher genus.

Proof: We will prove the inequality $\dim(\mathrm{V}\otimes \mathrm{I}_{n-1})^{\mathrm{H}_n} < \dim(\mathrm{I}_n)^{\mathrm{H}_n}$, which implies that the multiplication map $\mathrm{V}\otimes \mathrm{I}_{n-1} \to \mathrm{I}_n$ cannot be surjective. Let us treat first the case $n=3$. From the exact sequence $0 \to \mathrm{I}_3 \to \mathsf{S}^3\mathrm{V} \to \mathrm{H}^0(\mathrm{A}, \mathcal{L}^9) \to 0$ (2.1) we get

$$\dim \mathrm{I}_3 = \binom{\mathrm{N}+2}{3} - \mathrm{N}^2 = \frac{\mathrm{N}-3}{6}(\mathrm{N}^2-1) + \frac{\mathrm{N}-1}{2} \ ;$$

as in Proposition 1 we conclude that $\dim(\mathrm{I}_3)^{\mathrm{H}_3} = (\mathrm{N}-1)/2$.

Let $\mathrm{K} \subset \mathsf{S}^3\mathrm{V}$ be the space of H_3-invariant cubic forms singular along $\varphi_3(\mathrm{A})$; by the proposition the natural map $\mathrm{V}^*\otimes \mathrm{K} \to \mathrm{I}_2$ is an isomorphism. The action of H_3 on K is trivial, and the H_3-module $\mathrm{V}\otimes\mathrm{V}^*$ is the direct sum of a one-dimensional factor for each character of A_3 ; thus

$$\dim(\mathrm{V}\otimes \mathrm{I}_2)^{\mathrm{H}_3} = \dim \mathrm{K} = \frac{1}{2}(3^g - 2^{g+1} + 1) < \frac{1}{2}(\mathrm{N}-1)$$

(2.1), hence the result.

For $n=4$ the same method gives $\dim(\mathrm{I}_4)^{\mathrm{H}_2} = \frac{1}{6}(\mathrm{N}-1)(\mathrm{N}-2)$, which is larger than $\dim(\mathrm{V}\otimes \mathrm{I}_3)^{\mathrm{H}_2} = \frac{1}{6}(\mathrm{N}(\mathrm{N}+3) - 3^{g+1} - 1)$. ∎

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